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Graphs and Separability Properties of Groups

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A group G is LERF (locally extended residually finite) if for any finitely generated subgroup S of G and for any $g \notin S$ there exists a finite index subgroup S_0 of G which contains S but not g. Using graph-theoretical methods we give algorithms for constructing finite index subgroups in amalgamated free products of groups with good separability properties. We prove that a free product of a free group and a LERF group amalgamated over a cyclic subgroup maximal in the free factor is LERF. The maximality condition cannot be removed, because adjunction of roots does not preserve property LERF. We also give short proofs of some old theorems about separability properties of groups, including a theorem of Brunner, Burns, and Solitar that a free product of free groups amalgamated over a cyclic subgroup is LERF.

INTRODUCTION

A group G is RF (residually finite) if for any nontrivial element $g \in G$ there exists a finite index subgroup S_0 of G which does not contain g.

A group G is LERF (locally extended residually finite) if for any finitely generated subgroup S of G and for any $g \notin S$ there exists a finite index subgroup S_0 of G which contains S but not g.

The separability properties of groups have been an object of study for a long time, see [L-S], [A-G], [A-T], [We] for various results and additional references. RF and LERF groups have various interesting properties. For example, RF groups have a solvable word problem and LERF groups have a solvable generalised word problem, i.e., given a finite subset of the group there is an algorithm to find out if a given element belongs to the subgroup generated by that set.

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Fundamental groups of Haken 3-manifolds are RF, see [He]. If the fundamental group of a 3-manifold M is LERF, then for any surface subgroup K of the fundamental group of M there exist a finite cover $\tilde{M}$ of M and an incompressible surface N embedded in $\tilde{M}$ such that the fundamental group of N is K [Scott1]. Recall that an irreducible orientable 3-manifold which has an embedded incompressible surface is called Haken. The fundamental groups of Seifert Fibered Spaces are LERF [Scott2], and there exist compact 3-manifolds with non-LERF fundamental groups [B-K-S]. It is unknown if the fundamental groups of hyperbolic 3-manifolds are LERF. If they are, then any hyperbolic manifold which has a surface subgroup in its fundamental group is virtually Haken, i.e., can be finitely covered by a Haken manifold.

The study of LERF groups was initiated by M. Hall who proved that free groups are LERF [Hall]. Then it was shown that a free product of two LERF groups is LERF [Burns], [Ro], a free product of two LERF groups amalgamated over a finite subgroup is LERF [A-G], a free product of free groups with cyclic amalgamation is LERF [B-B-S], a free product of surface groups with cyclic amalgamation is LERF [Ni], and a free product of nilpotent groups amalgamated over a cyclic subgroup is LERF [Tang]. Alternative proofs of some of these results using graph-theoretical methods are presented in this paper and in its sequel [Gi2]. On the other hand, examples show that an amalgamated product of two LERF groups with cyclic amalgamation need not be LERF [G-R1], [L-N], [Ri].

The main result of his paper is the following:

THEOREM 4.4. *If F is a free group, $f \in F$ is not a proper power, G is LERF, and g is of infinite order in G, then $F \underset{f=g}{*} G$ is LERF.*

The condition on f cannot be removed because, as shown in [G-R1], adjunction of roots need not preserve the property LERF.

Theorem 4.4 implies that a manifold constructed by glueing a Seifert fibered space and a handlebody along an incompressible boundary annulus has a LERF group, as long as the group of the annulus is a maximal cyclic subgroup of the handlebody group.

Let X be a set, let $X^* = \{x, x^{-1} \mid x \in X\}$, and for $x \in X$ define $(x^{-1})^{-1} = x$. Recall that the Cayley graph of the group $G = \langle X \mid R \rangle$ is an oriented graph whose set of vertices is G and the set of edges is $G \times X^*$, such that an edge (g, x) begins at the vertex g and ends at the vertex gx.

DEFINITION 0.1. Let S be a subgroup of $G = \langle X \mid R \rangle$, and let G/S denote the set of right cosets of S in G. The relative Cayley graph of G with respect to S is an oriented graph whose vertices are the cosets G/S, the set of edges is $(G/S) \times X^*$, such that an edge (Sg, x) begins at the vertex Sg and ends at the vertex Sgx. We denote it Cayley(G, S). The basepoint of Cayley(G, S) is $S \cdot 1$. If we want to emphasise the basepoint, we use the notation Cayley$(G, S, S \cdot 1)$. Note that S acts on the Cayley graph of G by left multiplication, and Cayley(G, S) can be defined as the quotient of the Cayley graph of G by this action.

The definition implies that S has finite index in G if and only if the graph Cayley(G, S) has finitely many vertices. Theorem 1.1 (cf. [Scott1]) reduces the question of existence of subgroups of finite index in G to the existence of graphs with certain properties.

THEOREM 1.1. *Let $G = \langle X \mid R \rangle$ be a group.*

(1) *G is RF if and only if for any finite tree Γ which is a subgraph of the Cayley graph of G there exists a finite index subgroup S_0 of G such that Γ can be embedded in* Cayley(G, S_0).

(2) *G is LERF if and only if for any finitely generated subgroup S of G and for any finite connected subgraph $(\Gamma, S \cdot 1)$ of* Cayley(G, S) *there exists a finite index subgroup S_0 of G such that $(\Gamma, S \cdot 1)$ can be embedded in* Cayley$(G, S_0, S_0 \cdot 1)$.

So we need to develop tools for construction and recognition of relative Cayley graphs. The motivation comes from the theory of covering spaces. Let K be a 2-complex representing the group $G = \langle X \mid R \rangle$. The 1-skeleton of K is a wedge of $|X|$ oriented labeled circles such that for any $x \in X$ there is a unique circle labeled with x. The 2-skeleton of K consists of $|R|$ discs such that for any $r \in R$ there is a unique disc whose boundary represents the word r in the 1-skeleton of K. As we work with a fixed presentation of G, the complex K is also fixed. Then Cayley(G, S) is the 1-skeleton of the cover of the complex K corresponding to the subgroup S. Therefore we will call the relative Cayley graphs of G "the covers of G." Such a cover is finite if and only if it has a finite number of vertices, which happens if and only if S has finite index in G.

DEFINITION 0.2. Let $G = \langle X \mid R \rangle$ be a group. A labeling of a graph Γ by the set X is a function $\mathrm{Lab} : E(\Gamma) \to X^*$ such that

(1) $\mathrm{Lab}(\bar{e}) = (\mathrm{Lab}(e))^{-1}$ for any $e \in E(\Gamma)$,

(2) if $\mathrm{Lab}(e_1) = \mathrm{Lab}(e_2)$ and $\iota(e_1) = \iota(e_2)$, then $e_1 = e_2$ (see Section 1 for notation).

A graph with a labeling function is called a labeled graph.

Labeled graphs have an obvious projection map into the 1-skeleton of the complex K, so they can be viewed as a generalisation of relative Cayley graphs. Indeed, any subgraph of Cayley(G, S) is a labeled graph with the labeling function $\mathrm{Lab}(S \cdot g, x) = x$.

DEFINITION 0.3. Let $G = \langle X \mid R \rangle$ be a group. Denote the set of all words in X^* by $W(X)$, and denote the equality of two words by $\equiv$. The label of a path $p = e_1 e_2 \cdots e_n$ in a labeled graph Γ is a function $\mathrm{Lab}(p) \equiv \mathrm{Lab}(e_1)\mathrm{Lab}(e_2) \cdot \ldots \cdot \mathrm{Lab}(e_n) \in W(X)$. As usual, we identify the word $\mathrm{Lab}(p)$ with the corresponding element in G.

DEFINITION 0.4. We say that a graph Γ is G-based, if any path p in Γ with $\mathrm{Lab}(p) = 1$ is closed.

Lemma 1.5 shows that G-based labeled graphs are subgraphs of relative Cayley graphs, allowing us to reformulate Theorem 1.1.

THEOREM 1.1 (restated). *Let $G = \langle X \mid R \rangle$ be a group.*

(1) *G is RF if and only if any finite G-based tree labeled with X can be embedded in a cover of G with finitely many vertices.*

(2) *G is LERF if and only if any finite connected G-based graph labeled with X can be embedded in a cover of G with finitely many vertices.*

Let X and Y be sets such that $X^* \cap Y^* = \emptyset$ and let $A = G \underset{G_0 = H_0}{*} H$ be an amalgamated free product of LERF groups $G = \langle X \mid R \rangle$ and $H = \langle Y \mid T \rangle$. Let S be a finitely generated subgroup of $A = G \underset{G_0 = H_0}{*} H$ and let Γ be a finite subgraph of Cayley(A, S). Theorem 1.1 implies that A is LERF if for each such S and Γ there exists an embedding of Γ in a cover of A with finitely many vertices. We will attempt to construct such an embedding in two steps.

DEFINITION 0.5. Let X^* and Y^* be disjoint sets and let Γ be a graph labeled with $X \cup Y$. A subgraph of Γ is called monochromatic if it is labeled only with X or only with Y. An X-component of Γ is a maximal connected subgraph of Γ labeled with X and a Y-component of Γ is a maximal connected subgraph of Γ labeled with Y.

DEFINITION 0.6. Let $A = G \underset{G_0 = H_0}{*} H$. We say that an A-based graph Γ labeled with $X \cup Y$ is a precover of A if each X-component of Γ is a cover of G and each Y-component of Γ is a cover of H.

Step 1. Try to embed Γ in a precover Γ' of A with finitely many vertices.

We attempt to construct Γ' as follows. Let Γ_i, $1 \le i \le n$, be the set of all monochromatic components of Γ. If G and H are LERF, then for any $1 \le i \le n$ there exists an embedding f_i of Γ_i in a graph Γ'_i, which is a cover of G or of H with finitely many vertices. (If G and H are RF, Γ'_i exists as long as Γ is a tree.) Let Γ' be the graph obtained from Γ by replacing each Γ_i with Γ'_i (cf. Remark 2.3 for the exact definition of Γ').

This step is discussed in Section 4. We show that we can choose Γ'_i such that Γ' is a precover if G and H are free and G_0 is cyclic (cf. Lemma 4.5) or if G is LERF, H is free, and H_0 is a maximal cyclic subgroup of H (cf. Lemma 4.3).

Step 2. Try to show that any precover of A with finitely many vertices can be embedded in a cover of A with finitely many vertices.

Recall that a group H is residually (torsion-free nilpotent) if for any nontrivial element h of H there exists a homomorphism of H onto a torsion-free nilpotent group such that h does not lie in the kernel. We prove that the second step of the construction can always be completed if H is residually (torsion-free nilpotent) and LERF, G is LERF, and G_0 is infinite cyclic (cf. Theorem 3.7) or if both G and H are residually (torsion-free nilpotent) and G_0 is infinite cyclic (cf. Theorem 3.4).

A theorem of Magnus [Bo, p. 174] says that free groups are residually (torsion-free nilpotent), so Theorem 4.4 follows from Lemma 4.3 and Theorem 3.7.

In the sequel to this paper [Gi2] we examine separability properties of more general amalgamated free products. Similar methods are used in [G-R2] to study the separability properties of double cosets in a free group.

1. LABELED GRAPHS

We follow the notation and terminology of J.-P. Serre [Serre] and J. R. Stallings [Sta]. A graph Γ consists of two sets E and V, and two functions $E \to E$ and $E \to V$. The elements of E are called edges, and the elements of V are called vertices. For any $e \in E$ there is $\bar{e} \in E$ and $\iota(e) \in V$, such that $\bar{\bar{e}} = e$ and $\bar{e} \neq e$, $\iota(e)$ is the initial vertex of e, and $\tau(e) = \iota(\bar{e})$ is the terminal vertex of e. An orientation of Γ is a choice of exactly one edge in each pair $\{e, \bar{e}\}$. A path p of length $n = |p|$ in Γ is a finite sequence of edges $p = e_1 e_2 \cdots e_n$ with $\iota(e_j) = \tau(e_{j-1})$. The initial vertex of p is $\iota(e_1) = \iota(p)$ and the terminal vertex of p is $\tau(e_n) = \tau(p)$. The inverse of p is the path $\bar{p} = \bar{e}_n \bar{e}_{n-1} \cdots \bar{e}_1$. A path is closed if its initial and terminal vertices coincide. A path is reduced if it does not contain a subpath of the form $e\bar{e}$. A graph is connected if any pair of vertices is joined by some path. A graph is a tree if it is connected and if it does not contain a nontrivial reduced closed path. A graph is finite if it has finitely many edges and vertices. Denote the pair of the graph Γ and the basepoint v_0 by (Γ, v_0). A map of graphs consists of a pair of functions, which map edges to edges, vertices to vertices, commute with ι and τ, and preserve the basepoints.

THEOREM 1.1. *Let $G = \langle X \mid R \rangle$ be a group.*

(1) *G is RF if and only if for any finite tree Γ which is a subgraph of the Cayley graph of G there exists a finite index subgroup S_0 of G such that Γ can be embedded in* $\mathrm{Cayley}(G, S_0)$.

(2) *G is LERF if and only if for any finitely generated subgroup S of G and for any finite connected subgraph $(\Gamma, S \cdot 1)$ of* $\mathrm{Cayley}(G, S)$ *there exists a finite index subgroup S_0 of G such that $(\Gamma, S \cdot 1)$ can be embedded in* $\mathrm{Cayley}(G, S_0, S_0 \cdot 1)$.

Proof. We prove part (2) of the theorem. The proof of the first part is similar. Assume G is LERF. Let S be a finitely generated subgroup of G, and let $(\Gamma, S \cdot 1)$ be a finite connected subgraph of $\mathrm{Cayley}(G, S)$. For any $v_i \in V(\Gamma)$ choose a path p_i in Γ which begins at $S \cdot 1$ and ends at v_i. Denote $\mathrm{Lab}(p_i)$ by g_i. Then $g_i \cdot g_j^{-1} \notin S$ for $i \neq j$. As S is finitely generated and as G is LERF, there exists $S_0 <_f G$ such that $S \leq S_0$ and $g_i \cdot g_j^{-1} \notin S_0$ for $i \neq j$. Define a map $\rho : (\Gamma, S \cdot 1) \to \mathrm{Cayley}(G, S_0, S_0 \cdot 1)$ as follows: $\rho(v_i) = S_0 \cdot g_i$ for $v_i \in V(\Gamma)$ and $\rho(v_i, x) = (\rho(v_i), x)$ for $(v_i, x) \in E(\Gamma)$.

If $\rho(v_i) = \rho(v_j)$, then $S_0 \cdot g_i = S_0 \cdot g_j$, hence $i = j$. If $\rho(v_i, x_1) = \rho(v_j, x_2)$, then $v_i = v_j$ and $x_1 = x_2$, therefore ρ is the required embedding, proving one direction of the theorem.

To prove the other direction, let $S = \langle s_1, \cdots, s_m \rangle$ be a finitely generated subgroup of G, and let $g \notin S$. For each s_i choose one closed path p_i in $\mathrm{Cayley}(G, S)$ which begins at $S \cdot 1$ with $\mathrm{Lab}(p_i) = s_i$. Let p_g be a path in $\mathrm{Cayley}(G, S)$ which begins at $S \cdot 1$ with $\mathrm{Lab}(p_g) = g$, and let Γ be a subgraph of $\mathrm{Cayley}(G, S)$ consisting of all the vertices and all the edges of the paths p_i and p_g. As $(\Gamma, S \cdot 1)$ is a finite connected subgraph of $\mathrm{Cayley}(G, S)$, by assumption, it can be embedded into $\mathrm{Cayley}(G, S_0, S_0 \cdot 1)$, where S_0 is finite index subgroup of G. Then the images of the loops p_i in $\mathrm{Cayley}(G, S_0)$ are loops beginning at $S_0 \cdot 1$, and the image of the path p_g is not a closed path, so $g \notin S_0$, but $s_i \in S_0$, hence $S \leq S_0$, therefore G is LERF.

DEFINITION 1.2. Let $G = \langle X \mid R \rangle$ be a group, and let Γ be a graph labeled with X. Denote the set of all closed paths in Γ starting at v_0 by $\mathrm{Loop}(\Gamma, v_0)$, and denote the image of $\mathrm{Lab}(\mathrm{Loop}(\Gamma, v_0))$ in G by $\mathrm{Lab}(\Gamma, v_0)$. It is easy to see that $\mathrm{Lab}(\Gamma, v_0)$ is a subgroup of G.

Remark 1.3. Any path p in $\mathrm{Cayley}(G, S)$ which begins at $S \cdot 1$ must end at $S \cdot \mathrm{Lab}(p)$, so p is a closed path if and only if $\mathrm{Lab}(p) \in S$. Therefore, $\mathrm{Lab}(\mathrm{Cayley}(G, S, S \cdot 1)) = S$.

DEFINITION 1.4. Let $x \in X$ and $v \in V(\Gamma)$. Following [G-T] we say that Γ is x-saturated at v, if there exists $e \in E(\Gamma)$ with $\iota(e) = v$ and $\mathrm{Lab}(e) = x$. We say that Γ is X^*-saturated if it is x-saturated for any $x \in X^*$ at any $v \in V(\Gamma)$.

The following lemma gives a characterization of the subgraphs of the relative Cayley graphs.

LEMMA 1.5. *Let* $G = \langle X \mid R \rangle$ *be a group and let* (Γ, v_0) *be a graph labeled with* X. *Denote* $\mathrm{Lab}(\Gamma, v_0) = S$. *Then*

(a) Γ *is* G*-based if and only if it can be embedded in* $\mathrm{Cayley}(G, S, S \cdot 1)$,

(b) Γ *is* G*-based and* X^**-saturated if and only if it is isomorphic to* $\mathrm{Cayley}(G, S, S \cdot 1)$.

Proof. Define a map $\mu : (\Gamma, v_0) \to \mathrm{Cayley}(G, S, S \cdot 1)$ as follows: $\mu(v) = S \cdot \mathrm{Lab}(p)$, where p is any path connecting v_0 and v, and $\mu(e) = (\mu(\iota(e)), \mathrm{Lab}(e)), v \in V(\Gamma), e \in E(\Gamma)$.

One direction of the lemma follows from the definition of the relative Cayley graph, and the other direction follows from the properties of the function μ listed in the Claim below.

CLAIM. (1) μ *is well-defined*,

(2) *if* Γ *is* G*-based, then* μ *is injective*,

(3) *if* Γ *is* X^**-saturated, then* μ *is surjective*.

Proof of the Claim. (1) If p and q are paths in Γ with $\iota(p) = \iota(q) = v_0$ and $\tau(p) = \tau(q) = v$, then $\mathrm{Lab}(p) \cdot (\mathrm{Lab}(q))^{-1} \in \mathrm{Lab}(\Gamma, v_0) = S$, so $S \cdot \mathrm{Lab}(p) = S \cdot \mathrm{Lab}(q)$, therefore μ is well-defined.

(2) If $\mu(v_1) = \mu(v_2)$, then there are paths p and q in Γ with $\iota(p) = \iota(q) = v_0$, $\tau(p) = v_1$, $\tau(q) = v_2$, and $S \cdot \mathrm{Lab}(p) = S \cdot \mathrm{Lab}(q)$. But then there is a closed path $t \in \mathrm{Loop}(\Gamma, v_0)$ with $\mathrm{Lab}(p) = \mathrm{Lab}(t) \cdot \mathrm{Lab}(q)$, so $\mathrm{Lab}(\bar{p}tq) = 1$, and since Γ is G-based, $\iota(\bar{p}tq) = v_1 = v_2 = \tau(\bar{p}tq)$, so μ is injective on vertices. If $\mu(e_1) = \mu(e_2)$, then $\mu(\iota(e_1)) = \mu(\iota(e_2))$ and $\mathrm{Lab}(e_1) = \mathrm{Lab}(e_2)$, so $\iota(e_1) = \iota(e_2)$, and property (2) of the labeling function implies that $e_1 = e_2$, so μ is injective on edges.

(3) As Γ is X^*-saturated, for any $g \in G$ there is a path p in Γ with $\iota(p) = v_0$ and $\mathrm{Lab}(p) = g$. Then $\mu(\tau(p)) = S \cdot g$, so μ is surjective on vertices. For any edge (Sg, x) in $\mathrm{Cayley}(G, S)$ let $e \in E(\Gamma)$ be the edge with $\mu(\iota(e)) = Sg$, and $\mathrm{Lab}(e) = x$. Such e exists because Γ is X^*-saturated. Then $\mu(e) = (Sg, x)$, so μ is surjective on edges.

Remark 1.6. Let $F = \langle X \rangle$ be a free group. Any graph labeled with X is F-based, hence it is a subgraph of some cover of F, and any X^*-saturated graph labeled with X is a cover of F.

To illustrate applications of Lemma 1.5 we give a short proof of M. Hall's theorem.

COROLLARY 1.7. *Free groups are LERF* [*Hall*].

Proof. Let $F = \langle X \rangle$ be a free group and let Γ be a finite graph labeled with X. We embed Γ in a cover of F with finitely many vertices as follows. For any vertex $v \in V(\Gamma)$ and for any $x \in X$ the number of edges in Γ labeled with x which have an endpoint at v is either 0, 1, or 2. In the last case we do nothing. In the first case we add a loop labeled with x to the vertex v. In the second case we find the maximal path p in Γ which is labeled by a power of x and begins at v. By assumption p is not a loop. We add an edge e to Γ such that $\iota(e) = \tau(p)$ and $\tau(e) = v$. If p is labeled by a positive power of x, we label e with x. If p is labeled by a negative power of x, we label e with x^{-1}. The resulting graph is labeled with X, it is X^*-saturated, and it has finitely many vertices, hence Remark 1.6 implies that it is a cover of F, but then Theorem 1.1 (restated) implies that F is LERF.

Remark 1.8. The proof of Corollary 1.7 shows that any finite F-based graph Γ can be embedded into a cover of F without changing the set of vertices of Γ. So if $S = \langle s_1, \cdots, s_m \rangle$ is a finitely generated subgroup of F and $f \notin S$, there exists a finite index subgroup S_0 in F which contains S, but not f, and the index of S_0 in F is less than $|s_1| + \cdots + |s_m| + |f|$. This construction will be used in the proofs of Theorems 4.4 and 4.6.

2. PROPERTIES OF PRECOVERS

Following [Sta] we describe how to build new labeled groups out of old ones. The amalgam of labeled graphs Γ_1 and Γ_2 along Γ_0, denoted by $\Gamma_1 \underset{\Gamma_0}{*} \Gamma_2$, is the pushout of the following diagram in the category of labeled graphs.

$$\begin{array}{ccc} \Gamma_0 & \xrightarrow{i_1} & \Gamma_1 \\ {\scriptstyle i_2}\downarrow & & \\ \Gamma_2 & & \end{array}$$

where i_1 and i_2 are injective maps and none of the graphs need be connected. The amalgam depends on the maps i_1 and i_2, but we omit reference to them, whenever it does not cause confusion.

Remark 2.1. Let $G = \langle X \mid R \rangle$ be a group, let Γ be a graph, and let $f : E(\Gamma) \to X^*$ be a function such that $f(\bar{e}) = (f(e))^{-1}$ for any $e \in E(\Gamma)$. Let e_1 and e_2 be distinct edges of Γ such that $\iota(e_1) = \iota(e_2)$ and $f(e_1) =$

$f(e_2)$. Define Γ_1 to be the graph with $E(\Gamma_1) = E(\Gamma)/e_1 \sim e_2$ and $V(\Gamma_1) = V(\Gamma)/\tau(e_1) \sim \tau(e_2)$. The projection of Γ onto Γ_1 is called a folding of e_1 and e_2. The graph constructed by performing all possible foldings of Γ is a labeled graph. It can be easily seen that amalgamation consists of taking the disjoint union of graphs and performing the identifications prescribed by i_1 and i_2 and subsequent foldings until a labeled graph is obtained.

Remark 2.2. Note that Γ_1 and Γ_2 may not be embedded in their amalgam. For example, let Γ_1 consist of a single loop p_1 of length 3 with $\mathrm{Lab}(p_1) \equiv x^3$, let Γ_2 consist of a single loop p_2 of length 2 with $\mathrm{Lab}(p_2) \equiv x^2$, and let Γ_0 be a single vertex. Then $\Gamma_1 \underset{\Gamma_0}{*} \Gamma_2$ consists of a single loop of length 1 labeled with x.

Remark 2.3. Let S be a finitely generated subgroup of $A = G \underset{G_0 = H_0}{*} H$ and let Γ be a finite subgraph of Cayley(A, S). We want to embed Γ in a precover of A with finitely many vertices. Let Γ_i, $1 \le i \le n$, be the set of all monochromatic components of Γ. If G and H are LERF, then for any $1 \le i \le n$ there exists an embedding f_i of Γ_i in a graph Γ_i', which is a cover of G or of H with finitely many vertices. Let Γ' be the amalgam of Γ and all Γ_i' given by the diagram

$$\begin{array}{ccc} \sqcup\Gamma_i & \xrightarrow{\sqcup g_i} & \Gamma \\ {\scriptstyle \sqcup f_i}\Big\downarrow & & \Big\downarrow \\ \sqcup\Gamma_i' & \longrightarrow & \Gamma' \end{array}$$

where g_i is the inclusion map of Γ_i into Γ. (If G and H are RF, Γ_i' exists as long as Γ is a tree.) Note that as each Γ_i is embedded in Γ_i', and no foldings occur between distinct Γ_i', it follows that Γ is embedded in Γ'. If Γ' turns out to be A-based, which as Example 2.6 shows need not happen, then Lemma 1.5 asserts that Γ' can be embedded in some cover of A.

To illustrate this idea we give short proofs of some old theorems about separability properties of free products. Note that if $A = G * H$ is a free product, then any graph labeled with $X \cup Y$ is A-based if and only if each G-component is G-based and each H-component is H-based. In particular, it is A-based if each monochromatic component is a cover.

LEMMA 2.4. *For any groups $G = \langle X \mid R \rangle$ and $H = \langle Y \mid T \rangle$ any precover of $G * H$ can be embedded in a cover of $G * H$ with the same set of vertices.*

Proof. Let Γ_X denote the wedge of $|X|$ loops each labeled with a different element of X, and let Γ_Y denote the wedge of $|Y|$ loops each labeled with a different element of Y. Let Γ be a precover of $G * H$. To any monochromatic vertex in Γ colored with X glue a copy of Γ_Y and to any monochromatic vertex in Γ colored with Y glue a copy of Γ_X. Each monochromatic component of the resulting graph Γ' is a cover, so Γ' is A-based. It is $X^* \cup Y^*$-saturated, hence Lemma 1.5 implies that it is the required cover of $G * H$.

THEOREM 2.5. (1) *Theorem of Gruenberg*: *any free product of RF groups is RF* [*Gru*].

(2) *Theorem of Burns and Romanovskii*: *any free product of LERF groups is LERF* [*Bu*], [*Ro*].

Proof. Let G and H be groups, let $A = G * H$, and let Γ be a finite A-based graph. If G and H are LERF, or if G and H are RF and Γ is a tree, we can embed Γ in a graph Γ' as described in Remark 2.3. As explained above, Γ' is A-based, so it is a percover of A with finitely many vertices, hence Lemma 2.4 implies that it can be embedded in a cover of A with finitely many vertices. Therefore Theorem 1.1 (restated) implies the result.

EXAMPLE 2.6. Let G, G_0, H, and H_0 be infinite cyclic groups generated by x, x^2, y, and y^3, respectively. Then $A = G \underset{G_0 = H_0}{*} H$ is the fundamental group of the trefoil knot. Let Γ be a graph consisting of two edges e_1 and e_2 and three vertices v_0, v_1, and v_2 such that $\iota(e_1) = \iota(e_2) = v_0$, $\tau(e_1) = v_1$, and $\tau(e_2) = v_2$, and such that $\mathrm{Lab}(e_1) = x$, $\mathrm{Lab}(e_2) = y$. Then Γ has two monochromatic components: $\Gamma_1 = e_1$ and $\Gamma_2 = e_2$. Let Γ_1' and Γ_2' be loops of length 2 labeled with x^2 and y^2 correspondingly. Then Γ_1' is a finite cover of G, Γ_2' is a finite cover of H, and Γ_i embeds in Γ_i'. However, the graph Γ' given by the diagram in Remark 2.3 is not A-based. If we want to make it A-based, we have to identify the vertices v_0 and v_2 and then to fold the edges of Γ_2', so that the resulting graph Γ'' consists of a loop labeled with x^2 and a loop labeled with y. Unfortunately, the original graph Γ does not embed in Γ''.

In order to decide when the graph Γ' constructed in Remark 2.3 is A-based, we need more tools.

DEFINITION 2.7. Let $A = G *_{G_0 = H_0} H$. We call G and H "the factors of A." A word $a \equiv a_1 a_2 \cdots a_n \in A$ is in normal form if:

(1) a_i lies in one factor of A,

(2) a_i and a_{i+1} are in different factors of A,

(3) if $n \neq 1$, then $a_i \notin G_0$.

Any $a \in A$ has a representative in normal form. If $a \equiv a_1 a_2 \cdots a_n$ is in normal form and $n > 1$, then the Normal Form Theorem [L-S, p. 187] implies that $a \neq 1_A$.

DEFINITION 2.8. Let p be a path in a graph labeled with $X \cup Y$, and let $p_1 p_2 \cdots p_n$ be its decomposition into maximal monochromatic subpaths. We say that p is in normal form if $\text{Lab}(p) \equiv \text{Lab}(p_1) \cdots \text{Lab}(p_n)$ is in normal form.

DEFINITION 2.9. Let Γ be a graph labeled with $X \cup Y$. We say that a vertex $v \in V(\Gamma)$ is bichromatic if there exist edges e_1 and e_2 in Γ with $\iota(e_1) = \iota(e_2) = v$, $\text{Lab}(e_1) \in X^*$, and $\text{Lab}(e_2) \in Y^*$. We say that v is X-monochromatic if all the edges of Γ beginning at v are labeled with X and we say that v is Y-monochromatic if all the edges of Γ beginning at v are labeled with Y.

DEFINITION 2.10. We say that Γ is compatible at a bichromatic vertex v if for any monochromatic path p in Γ such that $\iota(p) = v$ and $\text{Lab}(p) \in G_0$ there exists a monochromatic path t of a different color in Γ such that $\iota(t) = v$, $\tau(t) = \tau(p)$, and $\text{Lab}(t) = \text{Lab}(p)$. We say that Γ is compatible if it is compatible at all bichromatic vertices.

Remark 2.11. Note that a precover is compatible. Indeed, let Γ be a precover and let p be a monochromatic path in Γ which begins at a bichromatic vertex v such that $\text{Lab}(p) \in G_0$. Without loss of generality p is labeled with X. As Γ is a precover, the Y-component of Γ containing v is a cover of H, hence it contains a path t which begins at v and has the same label as p. The path t is monochromatic labeled with Y. But $\text{Lab}(p\bar{t}) = 1$ and Γ is A-based, so $p\bar{t}$ is a closed path. Therefore $\tau(p) = \tau(t)$, so Γ is compatible.

LEMMA 2.12. *If Γ is a compatible graph, then for any path p in Γ there exists a path t in normal form which has the same endpoints and the same label as p.*

Proof. Let p be a path in Γ, and let $p_1 p_2 \cdots p_n$ be its decomposition into maximal monochromatic subpaths. The proof is by induction on the number n of the subpaths p_i in the above decomposition. If $n = 1$, then $p = p_1$ is in normal form, so it is the required path. Assume that the statement holds if the number of maximal monochromatic subpaths of p is less than n. If $\text{Lab}(p_i) \notin G_0$ for all $1 \le i \le n$, then p is in normal form. Otherwise, without loss of generality, assume that $\text{Lab}(p_j) \in G_0$ and p_j is labeled with X. Since one of the endpoints of p_j is bichromatic and Γ is compatible, there exists a path q_j in Γ labeled with Y with the same endpoints and the same label as p_j. Then the path $p^j = p_1 p_2 \cdots p_{j-1} q_j p_{j+1} \cdots p_n$ has the same endpoints and the same label as p. As $p_{j-1} q_j p_{j+1}$ is monochromatic, p^j has a decomposition into fewer than n monochromatic subpaths. Therefore, by the inductive hypothesis, there exists a path t in normal form which has the same endpoints and the same label as p^j. But then t has the same endpoints and the same label as p, and the inductive step is completed.

COROLLARY 2.13. *Let Γ be a compatible graph. If all G-components of Γ are G-based and all H-components of Γ are H-based, then Γ is A-based. In particular, if each G-component of Γ is a cover of G, each H-component of Γ is a cover of H, and Γ is compatible, then Γ is a precover of A.*

Proof. Let p be a path in Γ with $\mathrm{Lab}(p) = 1$. By Lemma 2.12 we may assume that p is in normal form. Then it follows from the Normal Form Theorem that $n = 1$ and p is monochromatic, hence p belongs to one monochromatic component of Γ. Since all X-components of Γ are G-based and all Y-components of Γ are H-based, p is closed, therefore Γ is A-based.

3. EMBEDDING PRECOVERS IN COVERS

Remark 3.1. Let H be a residually (torsion-free nilpotent) group. Then for any nontrivial h in H and for any $n \in N$ there exists a finite index normal subgroup H_n of H such that $H_n \cap \langle h \rangle = \langle h^n \rangle$ (cf. [Ste], [Gi1]).

DEFINITION 3.2. Let G be a group, let Γ be a G-based graph, and let G_0 be a subgroup of G. We say that the vertices v_1 and v_2 of Γ belong to the same G_0-orbit in Γ if Γ contains a path p such that $\iota(p) = v_1$, $\tau(p) = v_2$, and $\mathrm{Lab}(p) \in G_0$.

LEMMA 3.3. *Let $H = \langle Y \mid T \rangle$ be a residually (torsion-free nilpotent) group. Let g be an infinite order element in the group $G = \langle X \mid R \rangle$. Then any X^*-saturated precover of the group $A = G *_{g=h} H$ with finitely many vertices can be embedded in a cover of A with finitely many vertices.*

Proof. Let (Γ, v_0) be a X^*-saturated precover of A with finitely many vertices. Any vertex v of Γ has one of the two following types.

Type 1. Γ is saturated at v.

Type 2. v is X-monochromatic.

As Γ is a precover, it is compatible, so any $\langle g \rangle$-orbit in Γ consists of vertices of the same type. For any vertex v of the second type let n_v be the number of vertices in the $\langle g \rangle$-orbit of v in Γ. The proof is by induction on the number $n = n(\Gamma)$ which is the maximum of all n_v. Assume that Γ has m different $\langle g \rangle$-orbits, each containing n vertices of the second type. As mentioned in Remark 3.1, let H_n be a finite index normal subgroup of H such that $H_n \cap \langle h \rangle = \langle h^n \rangle$, and let k be the number of different $\langle h \rangle$-

orbits in Cayley(H, H_n). As H_n is normal in H, all $\langle h \rangle$-orbits in Cayley(H, H_n) contain n vertices. Let Γ_1 be the disjoint union of k isomorphic copies of Γ and let Γ_2 be the disjoint union of m isomorphic copies of Cayley(H, H_n). Then Γ_1 has $k \cdot m$ distinct $\langle g \rangle$-orbits, each containing n vertices of the second type, and Γ_2 has $k \cdot m$ distinct $\langle h \rangle$-orbits, each containing n vertices. Let $\{w_1, \cdots, w_{k \cdot m}\}$ be a set of representatives of these orbits and let $\{w_i^j, 0 \le j < n\}$ be the end-points of paths labeled with g^j which begin at w_i. Let $\{v_1, \cdots, v_{k \cdot m}\}$ be a set of representatives of all $\langle h \rangle$-orbits in Γ_2 and let $\{v_i^j, 0 \le j < n\}$ be the endpoints of paths labeled wtih h^j which begin at v_i. Let Γ'' be the amalgam of Γ_1 and Γ_2 over $k \cdot m \cdot n$ vertices, $\Gamma'' = \Gamma_1 \underset{\{w_i^j = v_i^j \mid 1 \le i \le k \cdot m,\, 0 \le j < n\}}{*} \Gamma_2$, and let Γ' be the connected component of Γ'' containing Γ. As all w_i^j are X-monochromatic and all v_i^j are Y-monochromatic, there cannot be any foldings between the images of Γ_1 and Γ_2 in Γ'', so Γ is embedded in Γ'. As Γ and Cayley(H, H_n) have finitely many vertices, so does Γ'. As the set $\{v_i^j, 1 \le i \le k \cdot m, 0 \le j < n\}$ is the set of all vertices of Γ_2 the graph Γ' is Y^*-saturated. By construction Γ' is compatible, so Corollary 2.13 implies that Γ' is a precover of A, but $n(\Gamma') < n(\Gamma)$, which completes the inductive step.

THEOREM 3.4. *Let G and H be residually (torsion-free nilpotent) groups and let g be an element of infinite order in G. Then any precover of the group $A = G \underset{g=h}{*} H$ with finitely many vertices can be embedded in a cover of A with finitely many vertices.*

Proof. First, using the fact that G is residually (torsion-free nilpotent), apply the construction described in Lemma 3.3 to embed any precover of A with finitely many vertices in an X^*-saturated precover of A with finitely many vertices. Then apply Lemma 3.3.

Remark 3.5. Let g be an element of infinite order in a group G. If G is LERF then for any integer n there exists a finite index subgroup G_n of G such that $g^n \in G_n$, but $g^i \notin G_n$ for $0 < i < n$. Then $G_n \cap \langle g \rangle = \langle g^n \rangle$.

LEMMA 3.6. *Let $G = \langle X \mid R \rangle$ be LERF, let h be an element of infinite order in $H = \langle Y \mid T \rangle$, and let Γ be a precover of $A = G \underset{g=h}{*} H$ with finitely many vertices. Then Γ can be embedded in an X^*-saturated precover of A with finitely many vertices.*

Proof. Any vertex v of Γ has one of the two following types.

Type 1. Γ is X^*-saturated at v.

Type 2. v is Y-monochromatic.

The proof is by induction on the number of vertices v of the second type. If no such vertices exist, then Γ is already X^*-saturated. Assume that Γ has m Y-monochromatic vertices, and let v be one of them. Let n be the integer such that $\mathrm{Lab}(\Gamma, v) \cap \langle h \rangle = \langle h^n \rangle$. Let $\{v_i \in V(\Gamma), 0 < i < n\}$ be the endpoints of paths labeled with h^i which begin at v_0. As explained in Remark 3.5, there exists a finite index subgroup G_n of G such that $G_n \cap \langle g \rangle = \langle g^n \rangle$.

Let $\Gamma' = \Gamma \underset{\{v_i = G_n \cdot g^i \mid 0 \le i < n\}}{*} \mathrm{Cayley}(G, G_n)$. As $\mathrm{Cayley}(G, G_n)$ is a cover of G it is labeled only with X, so all the vertices $G_n \cdot g^i$ are X-monochromatic. Also all v_i are Y-monochromatic. Indeed, if v_i belongs to an X-component Γ_i of Γ then, as Γ_i is a cover of G, it should contain a path p which begins at v_i labeled with g^{-i}. As Γ is a precover of A, the terminal vertex of p should be v, contradicting the assumption that v is Y-monochromatic. Hence there cannot be any foldings between the images of Γ and of $\mathrm{Cayley}(G, G_n)$ in Γ', so Γ is embedded in Γ'. As Γ and $\mathrm{Cayley}(G, G_n)$ have finitely many vertices, so does Γ'. Each monochromatic component of Γ' is a cover of G or of H, and by construction Γ' is compatible, hence Corollary 2.13 implies that Γ' is a precover. But the number of Y-monochromatic vertices in Γ' is less than m, which completes the inductive step.

THEOREM 3.7. *Let H be a residually (torsion-free nilpotent), let G be LERF, and let g be an infinite order element in G. Then any precover of the group $A = G \underset{g=h}{*} H$ with finitely many vertices can be embedded in a cover of A with finitely many vertices.*

Proof. This follows from Lemma 3.3 and Lemma 3.6.

4. CONSTRUCTING PRECOVERS

Recall that the degree of a vertex v in a graph Γ (denoted $\deg(v)$) is the number of edges of Γ beginning at v.

Remark 4.1. Let Γ be a (not necessarily connected) graph labeled with X. Let w_i^+ and w_i^-, $1 \le i \le n$, be distinct vertices of Γ such that $\deg(w_i^+) = \deg(w_i^-) = 1$, $1 \le i \le n$. Let e_i^+ be the (unique) edge of Γ which begins at w_i^+ and let e_i^- be the (unique) edge of Γ which ends at w_i^-. For each i, $1 \le i \le n$, add to Γ a reduced path t_i which intersects Γ only at its endpoints such that $\iota(t_i) = w_i^+$ and $\tau(t_i) = w_i^-$ and such that $t_i \cap t_j = \emptyset$, $i \ne j$. If the first edge of t_i and e_i^+ have different labels and if the last edge of t_i and e_i^- have different labels for all i, then the resulting graph Γ' is labeled with X.

LEMMA 4.2. *Let F_0 be a finitely generated subgroup of a free group $F = \langle X \rangle$, let f be a cyclically reduced element of F which is not a proper power, and let Γ be a finite subgraph of* $\text{Cayley}(F, F_0)$. *Let $\{w_1, \cdots, w_n\}$ be vertices of Γ which belong to different $\langle f \rangle$-orbits in* $\text{Cayley}(F, F_0)$ *such that* $\text{Lab}(\text{Cayley}(F, F_0, w_i)) \cap \langle f \rangle = \langle 1 \rangle$, $1 \leq i \leq n$. *For any $j > 0$, let p_i^{+j} be the reduced path in* $\text{Cayley}(F, F_0)$ *which begins at w_i and has the label f^j, and let p_i^{-j} be the reduced path in* $\text{Cayley}(F, F_0)$ *which begins at w_i and has the label f^{-j}. Let $w_i^{+j} = \tau(p_i^{+j})$, let $w_i^{-j} = \tau(p_i^{-j})$, and let Γ_j be the union of Γ with all p_i^{+j} and with all p_i^{-j}. There exists $N > 0$ such that for any $j > N$ and $1 \leq i \leq n$ the vertices w_i^{+j} and w_i^{-j} do not belong to Γ and have degree 1 in Γ_j.*

Proof. Recall that the core of a graph consists of all the vertices and all the edges of all reduced and cyclically reduced loops in the graph, hence the complement of the core is a union of trees. As F is free and F_0 is finitely generated, the core of $\text{Cayley}(F, F_0)$ is finite (cf. [Sta]). Let $N - 1$ be the number of vertices in the union of Γ and the core of $\text{Cayley}(F, F_0)$. As $\text{Lab}(\text{Cayley}(F, F_0, w_i)) \cap \langle f \rangle = \langle 1 \rangle$, $1 \leq i \leq n$, for any $j > N - 1$ and for $1 \leq i \leq n$ the vertices w_i^{+j} and w_i^{-j} do not belong to the union of Γ and the core of $\text{Cayley}(F, F_0)$.

We claim that the number N has the required properties. Indeed, otherwise there exists $j > N$ such that, without loss of generality, the degree of w_1^{+j} in Γ_j is bigger than 1. As f is cyclically reduced, for all k such that $j \geq k > 0$ and for all $1 \leq i \leq n$ the vertices w_i^{+k} belong to p_i^{+j}. As w_1^{+j} and $w_1^{+(j-1)}$ do not belong to the core of $\text{Cayley}(F, F_0)$, the definition of the core implies that, without loss of generality, they belong to p_2^{+j}. Assume that w_1^{+j} lies between vertices $w_2^{+(l+1)}$ and w_2^{+l} in p_2^{+j}. Then $w_1^{+(j-1)}$ lies between vertices w_2^{+l} and $w_2^{+(l-1)}$. Let f_1 be the label of the reduced path joining $w_2^{+(l-1)}$ to $w_1^{+(j-1)}$ and let f_2 be the label of the reduced path joining $w_1^{+(j-1)}$ to w_2^{+l}. Also let f_2 be the label of the reduced path joining w_2^{+l} to w_1^{+j} and let f_4 be the label of the reduced path joining w_1^{+j} to $w_2^{+(l+1)}$. Then $f_1 f_2 \equiv f_2 f_3 \equiv f$ and $|f| = |f_1| + |f_2| = |f_2| + |f_3|$, so $|f_1| = |f_3|$. Also $f_3 f_4 \equiv f$ and $|f| = |f_3| + |f_4|$, so f_1 and f_3 are initial subwords of f of equal length, hence $f_1 \equiv f_3$, therefore f_1 and f_2 commute. As F is free, the only commuting elements in F are powers of the same element, hence as $f \equiv f_1 f_2$ and $|f| = |f_1| + |f_2|$, f should be a proper power contradicting the choice of f. Therefore N has the required properties, proving the lemma.

LEMMA 4.3. *Let $f \in F = \langle X \rangle$ be not a proper power, let $H = \langle H \mid T \rangle$ be LERF, and let $A = F \underset{f=h}{*} H$. Then for any finitely generated subgroup S of A any finite connected subgraph $(\Gamma, S \cdot 1)$ of* $\text{Cayley}(A, S)$ *can be embedded in a precover of A with finitely many vertices.*

Proof. We can write $f = f_1 f_0 f_1^{-1}$, where f_0 is a cyclically reduced word. As A is isomorphic to $F \underset{f_0=h}{*} H$, we can assume that f is cyclically reduced. Let the number of distinct $\langle f\rangle$-orbits in Cayley(A, S) which contain a bichromatic vertex be m, and let $\{w_1, \cdots, w_m\}$ be a set of representatives of those orbits chosen such that all w_i are bichromatic. The required embedding is constructed in 3 steps.

Step 1. After renumbering, if needed, for k last values of the index i, $\mathrm{Lab}(\Gamma, w_i) \cap \langle f\rangle = \langle f^{n_i}\rangle \neq \langle 1\rangle$. For $m \geq i > m-k$ let p_i and q_i be monochromatic loops in Cayley(A, S) which begin at w_i such that $\mathrm{Lab}(p_i) \equiv f^{n_i}$ and $\mathrm{Lab}(q_i) \equiv h^{n_i}$. For any i such that $1 \leq i \leq m-k$, $\mathrm{Lab}(\Gamma, w_i) \cap \langle f\rangle = \langle 1\rangle$. For those values of i, let p_i^{+j} and p_i^{-j} be monochromatic paths labeled with X such that $\iota(p_i^{+j}) = \iota(p_i^{-j}) = w_i$, $\mathrm{Lab}(p_i^{+j}) = f^j$, $\mathrm{Lab}(p_i^{-j}) = f^{-j}$, for some $j > 0$, and let q_i^{+j} and q_i^{-j} be monochromatic paths labeled with Y such that $\iota(q_i^{+j}) = \iota(q_i^{-j}) = w_i$, $\mathrm{Lab}(q_i^{-j}) = \mathrm{Lab}(p_i^{-j})$ and $\mathrm{Lab}(q_i^{+j}) = \mathrm{Lab}(p_i^{+j})$. Let Γ_j be the union of Γ with all the paths p_i and q_i for $m-k < i \leq m$, and with all the paths $p^j +_i$, q_i^{+j}, p_i^{-j}, and q_i^{-j} for $1 \leq i \leq m-k$. Lemma 4.2 implies that there exists an integer N such that for all $j \geq N$ the vertices $w_i^{+j} = \tau(p_i^{+j})$ and $w_i^{-j} = \tau(p_i^{-j})$ have degree 1 in the X-component of Γ_j containing them, $w_i^{+j} \notin \Gamma$, and $w_i^{-j} \notin \Gamma$.

Step 2. Let N be as in the first step of the construction. As H is LERF and Γ_N is a finite graph, Theorem 1.1 implies that for any Y-component Γ_i^H of Γ_N there exists an embedding $\gamma_i : \sqcup\, \Gamma_i^H \to \tilde{\Gamma}_i^H$, where $\tilde{\Gamma}_i^H$ is a cover of H with finitely many vertices. Let Γ^* be the amalgam of Γ_N with $\sqcup \tilde{\Gamma}_i^H$ given by the diagram

$$\begin{array}{ccc} \sqcup\, \Gamma_i^H & \xrightarrow{\sqcup\, \alpha_i} & \Gamma_N \\ {\scriptstyle \sqcup\, \gamma_i}\Big\downarrow & & \Big\downarrow \\ \sqcup\, \tilde{\Gamma}_i^H & \longrightarrow & \Gamma^* \end{array}$$

where α_i is the inclusion map of Γ_i^H into Γ_N. As no foldings between the edges of Γ_N and $\tilde{\Gamma}_i^H$ are possible, Γ_N is embedded in Γ^*, and each Y-component of Γ^* is a cover of H with finitely many vertices. Also the sets of bichromatic vertices of Γ_N and Γ^* coincide. As each $\tilde{\Gamma}_i^H$ is a cover of H with finitely many vertices, for any vertex w_i^{+N} defined in the first step of the construction there exists a vertex w_j^{-N}, $1 \leq i,\ j \leq m-k$, a number $l_i > 0$, and a path s_i labeled with Y connecting the images of these vertices in Γ^* such that $\mathrm{Lab}(s_i) \equiv h^{l_i}$. As the images of w_i^{+N} and of w_j^{-N} have degree 1 in the X-component of Γ^* containing them, Remark 4.1 implies that for $1 \leq i \leq m-k$ we can add to Γ^* a set of disjoint paths

t_i labeled with X with $\mathrm{Lab}(t_i) = f^{l_i}$ such that t_i connects w_i^{+N} with the corresponding w_j^{-N} and the resulting graph is labeled. Identify all the corresponding vertices in the $\langle f \rangle$-orbit of the vertex w_i^{+N} in s_i and in t_i, $1 \leq i \leq m - k$. The resulting graph Γ^{**} is labeled with $X \cup Y$ and it is compatible, all its Y-components are covers of H, and all its X-components are F-based, hence by Corollary 2.13 it is A-based.

Step 3. Remark 1.8 implies that each X-component of Γ^{**} can be embedded in a cover of F with the same set of vertices. Let Γ' be the amalgam of Γ^{**} with all those covers given by the diagram similar to one in the second step of the construction. Then Γ' is compatible and each monochromatic component of Γ' is a cover, so Corollary 2.13 implies that Γ' is a precover of A. As Γ^{**} and Γ' have the same set of vertices, Γ' is a precover of A with finitely many vertices, as required.

THEOREM 4.4. *Let F be a free group and let G be LERF. If $f \in F$ is not a proper power, then $F \underset{f=g}{*} G$ is LERF.*

Proof. A theorem of Magnus [Bo, p. 174] says that free groups are residually (torsion-free nilpotent), so Theorem 4.4 follows from Lemma 4.3 and Theorem 3.7.

LEMMA 4.5. *Let $F = \langle X \rangle$ and $H = \langle Y \rangle$ be free groups and let $A = F \underset{f=h}{*} H$. Then for any finitely generated subgroup S of A, any finite connected subgraph* $(\Gamma, S \cdot 1)$ *of* $\mathrm{Cayley}(A, S)$ *can be embedded in a precover of A with finitely many vertices.*

Proof. Let f_0 and g_0 be the shortest roots of f and h. Say $f = f_0^a$ and $h = h_0^b$. As in the proof of Lemma 4.3, we can assume that f_0 and h_0 are cyclically reduced. The required embedding is constructed in 3 steps.

Step 1. This step is similar to the first step in the proof of Lemma 4.3, with few modifications. Let $\{w_i \mid 1 \leq i \leq m\}$ and $\{p_i, q_i \mid m - k < i \leq m\}$ be as in the proof of Lemma 4.3. Let W_1 be a set of representatives of distinct $\langle f_0 \rangle$-orbits of the set $\{w_i \mid 1 \leq i \leq m - k\}$ in the X-component of Γ, and let W_2 be a set of representatives of distinct $\langle h_0 \rangle$-orbits of the set $\{w_i \mid 1 \leq i \leq m - k\}$ in the Y-component of Γ. For $w_i \in W_1$ define p_i^{+j}, p_i^{-j}, w_i^{+j}, and w_i^{-j} as in Lemma 4.3. For $w_i \in W_2$ let t_i^{+j} and t_i^{-j} be monochromatic paths labeled with Y such that $\iota(t_i^{+j}) = \iota(t_i^{-j}) = w_i$, $\mathrm{Lab}(t_i^{+j}) = h^j$ and $\mathrm{Lab}(t_i^{-j}) = h^{-j}$, for some $j > 0$. Let Γ_j be the union of Γ with all the paths p_i and q_i for $m - k < i \leq m$, and with all the paths p_i^{+j}, p_i^{-j}, t_i^{+j}, t_i^{-j}. Lemma 4.2 implies that there exists an integer N_1 such that for all $j \geq N_1$ the vertices $w_i^{+j} = \tau(p_i^{+j})$ and $w_i^{-j} = \tau(p_i^{-j})$ have degree 1 in the X-component of Γ_j containing them, $w_i^{+j} \notin \Gamma$, and $w_i^{-j} \notin \Gamma$. Lemma 4.2 also implies that there exists an integer N_2 such that for all $j \geq N_2$ the vertices $u_i^{+j} = \tau(t_i^{+j})$ and $u_i^{-j} = \tau(t_i^{-j})$ have degree 1 in the Y-component of Γ_j containing them, $u_i^{+j} \notin \Gamma$, and $u_i^{-j} \notin \Gamma$.

Step 2. Let $N = \max\{N_1, N_2\}$. For each pair of vertices w_i^{+N} and w_i^{-N} add to Γ_N a path s_i labeled with X such that $\mathrm{Lab}(s_i) = f$, $\iota(s_i) = w_i^{+N}$, and $\tau(s_i) = w_i^{-N}$. For each pair of vertices u_i^{+N} and u_i^{-N} add to Γ_N a path t_i labeled with Y such that $\mathrm{Lab}(t_i) = h$, $\iota(t_i) = u_i^{+N}$, and $\tau(t_i) = u_i^{-N}$. As w_i^{+N} and w_i^{-N} have degree 1 in the X-component of Γ_N containing them, and u_i^{+N} and u_i^{-N} have degree 1 in the Y-component of Γ_N containing them, Lemma 4.1 implies that the resulting graph Γ^* is labeled with $X \cup Y$. By construction, any vertex w_i for $1 \leq i \leq m-k$ has $2N+1$ vertices in its $\langle f \rangle$-orbit in the X-component of Γ^* containing it and in its $\langle h \rangle$-orbit in the Y-component of Γ^* containing it. Identify corresponding vertices in the $\langle f \rangle$-orbit and in the $\langle h \rangle$-orbit of all w_i, $1 \leq i \leq m-k$. It is easy to see that the resulting graph Γ^{**} is compatible at any w_i, hence by Corollary 2.13, Γ^{**} is A-based.

Step 3. Remark 1.8 implies that each X-component of Γ^{**} can be embedded in a cover of F with the same set of vertices, and each Y-component of Γ^{**} can be embedded in a cover of H with the same set of vertices. Let Γ' be the amalgam of Γ^{**} with all those covers given by the diagram similar to one in the second step of the proof of Lemma 4.3. As no foldings between the edges of those covers and the edges of Γ^{**} are possible, Γ^{**} is embedded in Γ'. As Γ' is compatible and each monochromatic component of Γ' is a cover, Corollary 2.13 implies that Γ' is a precover of A. As Γ^{**} and Γ' have the same set of vertices, Γ' is a precover of A with finitely many vertices, as required.

THEOREM 4.6 (Theorem of Brunner, Burns, and Solitar). *A free product of free groups amalgamated over a cyclic subgroup is LERF* [*B-B-S*].

Proof. As free groups are residually (torsion-free nilpotent) the result follows from Theorem 3.4 and Lemma 4.5.

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REFERENCES

[A-G] R. B. J. T. Allenby and R. J. Gregorac, On locally extended residually finite groups, *in* "Conference on Group Theory," pp. 50–52, Lecture Notes in Math., Vol. 319, Springer-Verlag, New York/Berlin, 1973.

[A-T] R. B. J. T. Allenby and C. Y. Tang, The residual finiteness of some one-realtor groups with torsion, *J. Algebra* **71** (1981), 132–140.

[Ba] G. Baumslag, On the residual finiteness of generalized free produce of nilpotent groups, *Trans. Amer. Math. Soc.* **106** (1963), 193–209.

[B-E] J. Boler and B. Evans, The free product of residually finite groups amalgamated along retracts is residually finite, *Proc. Amer. Math. Soc.* **37,** (1973), 50–52.

[Bo] N. Bourbaki, "Groups et Algebres de Lie," Hermann, Paris, 1971.

[B-B-S] A. M. Brunner, R. G. Burns, and D. Solitar, The subgroup separability of free products of two free groups with cyclic amalgamation, *in* "Contemporary Mathematics," Vol. 33, pp. 90–115, Amer. Math. Soc., Providence, Rhode Island, 1984.

[Bu] R. G. Burns, On finitely generated subgroups of free products, *J. Austral. Math. Soc.* **12** (1971), 358–364.

[B-K-S] R. G. Burns, A. Karrass, and D. Solitar, A note on groups with separable finitely generated subgroups, *Bull. Austral. Math. Soc.* **36** (1987), 153–160.

[Gi1] R. Gitik, "Graphs, LERF Groups and Topology of 3-Manifolds," Ph.D. Thesis, Hebrew University, Jerusalem, 1990.

[Gi2] R. Gitik, Separability properties of doubles of groups, MSRI preprint No. 1996-048.

[G-R-1] R. Gitik and E. Rips, A necessary condition for $A \underset{a=b}{*} B$ to be LERF, *Israel J. Math.* **73** (1991), 125–127.

[G-R2] R. Gitik and E. Rips, On separability properties of groups, *Internat. J. Algebra Comput.* **5** (1995), 703–717.

[G-T] J. L. Gross and T. W. Tucker, "Topological Graph Theory," Wiley–Interscience, New York, 1987.

[Gru] K. W. Gruenberg, Residual properties of infinite soluble groups, *Proc. London Math. Soc.* **7** (1957), 29–62.

[Hall] M. Hall, Jr., Coset representations in free groups, *Trans. Amer. Math. Soc.* **67** (1949), 431–451.

[He] J. Hempel, Residual finiteness for 3-manifold groups, *in* "Combinatorial Group Theory and Topology," pp. 379–396, Ann. of Math. Stud., Vol. 111, Princeton Univ. Press, Princeton, New Jersey, 1987.

[L-N] D. D. Long and G. A. Niblo, Subgroup separability and 3-manifold groups, *Math. Z.* **207** (1991), 209–215.

[L-S] R. C. Lyndon and P. E. Schupp, "Combinatorial Group Theory," Springer-Verlag, New York/Berlin, 1977.

[Ni] G. A. Niblo, Fuchsian groups are strongly subgroup separable, preprint.

[Ri] E. Rips, An example of a non-LERF group, which is a free product of LERF groups with an amalgamated cyclic subgroup, *Israel J. Math.* **70** (1990), 104–110.

[Ro] N. S. Romanovskii, On the finite residuality of free products, relative to an embedding, *Izv. Akad. Nauk SSSR Ser. Mat.* **33** (1969), 1324–1329.

[Scott1] P. Scott, Subgroups of surface groups are almost geometric, *J. London Math. Soc.* **17** (1978), 555–565.

[Scott2] P. Scott, Correction to 'subgroups of surface groups are almost geometric', *J. London Math. Soc.* **32** (1985), 217–220.

[Serre] J.-P. Serre, "Trees," Springer-Verlag, New York/Berlin, 1980.

[Sta] J. R. Stallings, Topology of finite graphs, *Invent. Math.* **71** (1983), 551–565.

[Ste] P. F. Stebe, Conjugacy separability of certain free products with amalgamation, *Trans. Amer. Math. Soc.* **156** (1971), 119–129.

[Tang] C. Y. Tang, On the subgroup separability of generalised free products of nilpotent groups, *Proc. Amer. Math. Soc.* **113** (1991), 313–318.

[We] B. A. F. Wehfritz, The residual finiteness of some generalised free products, *J. London Math. Soc.* **24** (1981), 123–126.